§5 $\boxed{\text{Synchronization}}$

- Experiment: metronomes
- Active oscillators:
  - Van-der-Pol:

    $$m\ddot{x} - \gamma(\underbrace{1 - x^2}_{\text{negative friction}})\dot{x} + bx = 0.$$

  - Hopf

    $$\dot{z} = i\omega_0 z + \mu(1 - |z|^2)z.$$

  - Phase oscillator

    $$\dot{\varphi} = \omega_0.$$

Van-der Pol (working at Philips)

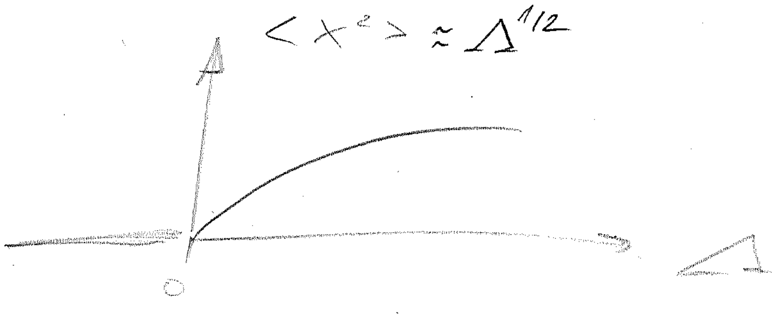
electrical circuits with
vacuum tube amplifiers

$\longrightarrow$ stable oscillator. [Nature 1927]

Later adaptation

- neuron models (FitzHugh–Nagumo)
- Seismology, geological faults.

Hopf bifurcation



$\longrightarrow$ map on Hopf normal form

$$0 = m\ddot{x} + \gamma(\cancel{\tfrac{1}{4}A} - x^2)\dot{x} + kx.$$

$y = \dot{x}$ $\quad , \omega_0 = \sqrt{\frac{k}{m}}$

- Idea $\quad Z \approx x - \frac{i}{\omega_0} y$

General form $\quad Z = x - \frac{i}{\omega} y + \sum_{k=2}^{\infty} \sum_{e=0}^{k} d_{k,e}\, x^e y^{k-e}.$

Ansatz: $\quad Z = x - \frac{i}{\omega_0} y +$

Back transformation $\quad + d_{10} y + d_{33} x^3 + d_{32} x^2 y + d_{31} x y^2 + d_{30} y^3. \quad \Rightarrow$ 5 parameters.

$$x = \underbrace{\frac{Z + \bar{Z}}{2}} + e_1 Z^3 + e_2 Z^2 \bar{Z} + e_3 Z \bar{Z}^2 + e_4 \bar{Z}^3 + \dots$$

$$y = \underbrace{i\omega \frac{Z - \bar{Z}}{2}} + f_1 Z^3 + f_2 Z^2 \bar{Z} + f_3 Z \bar{Z} + f_4 \bar{Z}^3.$$

$$\Rightarrow \dot{Z} = h(Z, \bar{Z})$$

$$= F Z + G Z \bar{Z}^2 + \text{h.o.t.}$$

- No quadratic terms $\Rightarrow$ no quadratic terms in Ansatz
- no term $\sim \bar{Z} \Rightarrow$ proper choice of d_{10}
- no terms proportional to $Z^3, Z^2\bar{Z}, \bar{Z}^3 \Rightarrow$ proper choice of $d_{3,e}$

⚠ Singularity of linear equation System prevents choice of $d_{3,e}$ with $G = 0$.

$$F = i\omega_0 + \frac{\gamma}{8m}\Lambda + O(\Lambda^2)$$

$$\sigma = \frac{\gamma}{8m} + O(\Lambda^2).$$

$$\dot{z} = i(\omega_c - \omega_1|z|^2)z + \mu(\Lambda - |z|^2)z.$$

$$\omega_c = \omega_0, \qquad \omega_1 = O(\Lambda)^2.$$

$$\mu = +\frac{\gamma}{8m}.$$

unit ob

m [kg].

k $[N/m] = [kg/s^2]$

γ $[N \frac{s}{m} \frac{1}{m^2}] = [\frac{kg}{s\,m^2}]$

μ $[\frac{1}{s\,m^2}]$.

$\mu\Lambda$ $[\frac{1}{s}]$

Λ $[m^2]$.

Hopf oscillator with noise.

$$\dot{Z} = i\,\omega_0 Z + \mu\,(A_0^2 - |Z|^2)\,Z + (i\,\xi_\varphi + \xi_A)\,Z \,.$$

$$\langle \xi_\varphi(t)\,\xi_\varphi(t')\rangle = 2D_\varphi\,.\,\delta(t-t')$$
$$\langle \xi_A(t)\,\xi_A(t')\rangle = 2D_A\;\delta(t-t')$$

Mapping on phase + amplitude

$$Z = A\cdot \exp i\varphi$$

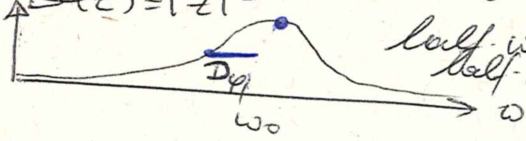
$$\left(\frac{\dot{A}}{A} + i\dot{\varphi}\right) Z = \dot{Z} = \ldots \qquad \Big/ Z$$

$$\frac{\dot{A}}{A} + i\dot{\varphi} = i\omega_0 + \mu\,(A_0^2 - A^2) + i\,\xi_\varphi + \xi_A \quad \Big/ \mathrm{Re}, \mathrm{Im}$$

- Phase correlation function
  $C(T) = \langle \exp i\varphi(t_0)\,\exp[i\varphi(t_0+T)]\rangle$.
  $|C(T)| = \exp(-D_\varphi |T|)$.

(*) $\dot{\varphi} = \omega_0 + \xi_\varphi$

- Quality factor $Q = \omega_0/(2D_\varphi)$



half-width-at-half-maximum.

(**) $A = A_0 + a$

$$\dot{a} = \mu\,(A_0 + a)\,(-2a\,A_0 + a^2) + \xi_A \,.$$
$$= -\,2\mu A_0\, a + \xi_A + O(a^2)$$

Ornstein-Uhlenbeck process.

$\langle a(t)\rangle = 0$.

$$\langle a(t)\,a(t')\rangle = \frac{D_A T}{2\mu A_0}\,\exp\left(-\frac{|t-t'|}{T}\right), \quad T = \frac{1}{2\mu A_0}$$

1d

N.B.
- isochronous
- non-isochronous $\dot{Z} = i(\omega_0 + |Z|^2\omega_1)Z + \ldots$

⇒ noisy phase oscillator: $\dot{\varphi} = \omega_0 + \xi$.

Two coupled oscillators.

$$\dot{\varphi}_L = \omega_L + C(\varphi_L - \varphi_R)$$
$$\dot{\varphi}_R = \omega_R + C(\varphi_R - \varphi_L).$$

$$\delta = \varphi_L - \varphi_R$$

⇒ $C(\delta) = C(\delta + 2\pi) \equiv$ coupling function

$$C(\delta) = \sum_n C_n' \cos(n\delta) + C_n'' \sin(n\delta).$$

vanish for $C(\delta) - C(-\delta)$

⇒

$$\dot{\delta} = \Delta\omega + 2 \sum_n C_n'' \sin \cancel{2} n\delta$$

⇒ only odd coupling terms contribute to synchronization.

Often C dominated by first Fourier mode.

$$\boxed{\dot{\delta} = \Delta\omega - \lambda \sin \delta}$$

$\equiv$ Adler equation ($\lambda = -2C_1''$).

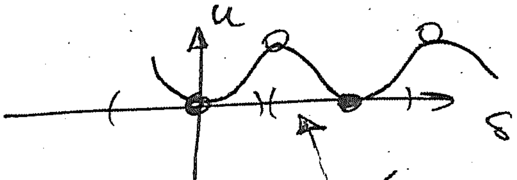
- fixed points:

  If $\Delta\omega = 0$: $\delta^* = 0$, $\delta^* = \pi$.

  generally, $\delta^* = \sin^{-1}(\Delta\omega/\lambda)$ if $|\Delta\omega| < |\lambda|$

- Stability? $\underbrace{\gamma \dot{\delta} = - \frac{\partial u}{\partial \delta}}_{\text{over-damped motion}}$, $\underbrace{u = -\gamma \Delta\omega\, \delta - \gamma \lambda \cos \delta}_{\text{eff. potential}}$.

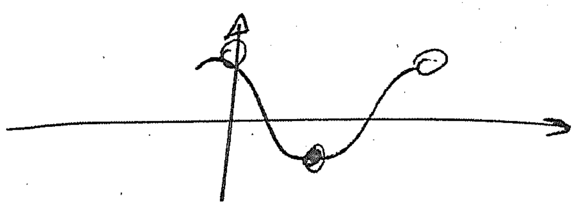
$\Delta\omega = 0$, $\lambda > 0$



basin of attraction.

in-phase synch.

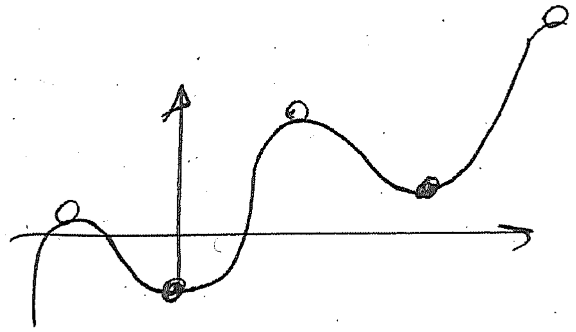
$\Delta\omega = 0$, $\lambda < 0$



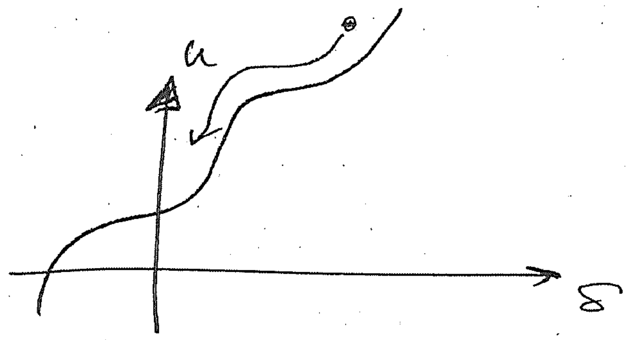
anti-phase synch.

$0 < \Delta\omega < \lambda$

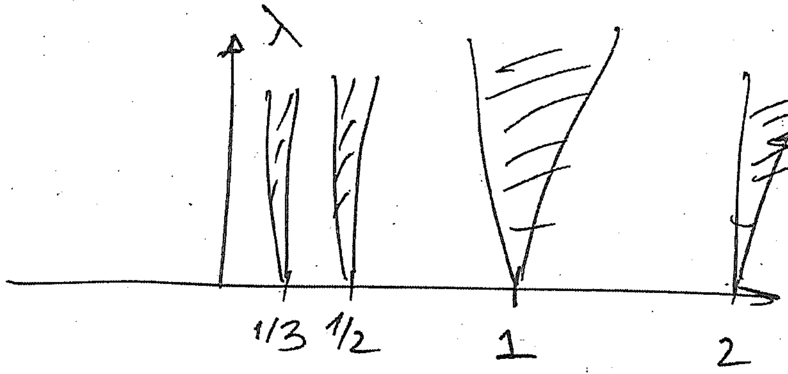
phase-lag between both oscillators.



$|\Delta\omega| > \lambda$: phase-drift (no synchr.).



- If $\omega_L : \omega_R \approx n:m$, $n, m \in \mathbb{N}$. there can be n:m-synchronisation



Synchronization in the presencence of noise

$$\dot{\delta} = \Delta\omega - \lambda \sin\delta + \xi .$$

$\xi(t) \equiv$ Gaussian white noise.

$\langle \xi(t) \rangle = 0$

$\langle \xi(t)\,\xi(t') \rangle = 2D\,\delta(t-t')$

$\longrightarrow$ 4a

$[D] = \frac{1}{s}$

Two coupled noisy oscillators.

Adding two noise terms

$$- \tfrac{\lambda}{2} \sin(\varphi_1 - \varphi_2)$$

$$\dot{\varphi}_1 = f_1(\varphi_1 - \varphi_2) + \xi_1(t)$$

$$\dot{\varphi}_2 = f_2(\varphi_1 - \varphi_2) + \xi_2(t)$$

$$\langle \xi_i(t)\, \xi_j(t') \rangle = \delta_{ij}\, \delta(t - t')\, 2D_i$$

$$\delta = \varphi_1 - \varphi_2$$

$$\dot{\delta} = f(\delta) + \underbrace{\xi_1(t) + \xi_2(t)}_{\xi(t)}$$

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(t)\, \xi(t') \rangle = 2D_1\, \delta(t - t') + 2D_2\, \delta(t - t') + 0.$$

$\longrightarrow$

$\xi(t) \equiv$ gaussian white noise with noise strength $2(D_1 + D_2)$

4a

$$\gamma \dot{\delta} = - \frac{\partial u}{\partial \delta} + \xi .$$

$\xi(t) \equiv$ gaussian white noise

$\langle \xi(t) \rangle = 0.$

$\langle \xi(t)\, \xi(t') \rangle = 2D' \delta(t-t').$

$$D' = \frac{kT_{eff}}{\gamma} \gamma^2 .$$

formally similar to ~~temperature~~, yet can be due to active processes.

What is the effect of noise?

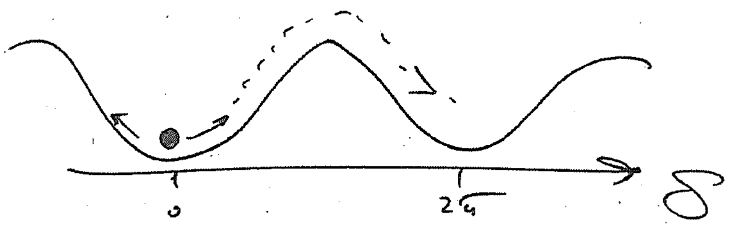
$\Rightarrow$ synchronization perturbed

- δ fluctuates around δ^*
- occasionally phase slips: $\delta \to \delta \pm 2\pi$.

⇒ Steady-state probability distribution formally equiv. to Boltzmann distrib.

$$p(\delta) \sim \exp\left(-\frac{U(\delta)}{kT_{eff}}\right) \quad \text{for } \Delta\omega = 0.$$

$$= \frac{1}{2\pi I_0(\lambda/D)} \exp\left(\frac{\lambda}{D}\cos\delta\right).$$



phase-slips:

$\delta \longrightarrow \delta + 2\pi$ with rate G_+

$\delta \longrightarrow \delta - 2\pi$: — " — G_-

$$G_\pm = \frac{D}{4\pi^2} \left| I_{i\Delta\omega/D}(\lambda/D) \right|^{-2} \exp\left(\pm \frac{\Delta\omega}{D}\right)$$

⟶ Mathematica-notebook

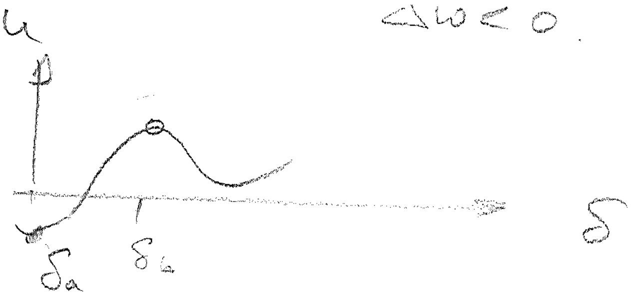
For $\Delta\omega = 0$:

$$G_+ = G_- = \begin{cases} \exp(-2\lambda/D)/(2\pi) & D \ll |\lambda| \\ D/(2\pi)^2 & D \gg |\lambda|. \end{cases}$$

⟶ Stratonovich

• $\Delta\omega \lesssim \lambda$ ⇒ giant diffusion [Hänggi]

$\Delta\omega < 0.$



$$1/T_a = U''|_{\delta=\delta_a} = \sqrt{\lambda^2 - \Delta\omega^2},$$

$$1/T_b = -U''|_{\delta=\delta_b} = 1/T_a$$

$$\sigma_+ = 2\pi\, T_a \cdot \exp\left(\frac{\Delta E}{D}\right)$$

$$\Delta E = U(\delta_b) - U(\delta_a).$$

σ_- ... analogously

- $$\frac{\sigma_+}{\sigma_-} = \exp(-2\pi\, \Delta\omega / D).$$

- for $\Delta\omega = 0.$

$$\sigma_+ = \sigma_- = \frac{\lambda}{2\pi} \exp\left(-\frac{2\lambda}{D}\right).$$

Phase-slip rate from Kramer's theory.

In[102]:=
```
U[δ_] := λ Cos[δ] + Δω δ
```

In[103]:=
```
dU = D[U[δ], δ]
ddU = D[U[δ], {δ, 2}]
```

Out[103]= $\Delta\omega - \lambda \operatorname{Sin}[\delta]$

Out[104]= $-\lambda \operatorname{Cos}[\delta]$

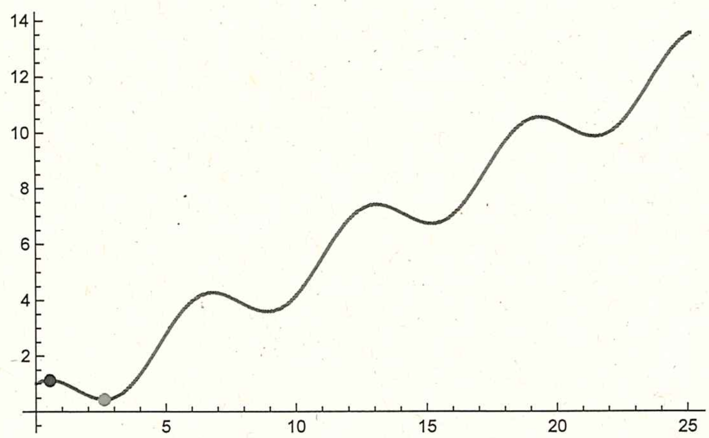
In[105]:=
```
Solve[dU == 0, δ]
```

Out[105]= $\left\{\left\{\delta \to \text{ConditionalExpression}\left[\pi - \text{ArcSin}\left[\frac{\Delta\omega}{\lambda}\right] + 2\pi\, C[1],\ C[1] \in \text{Integers}\right]\right\},\right.$
$\left.\left\{\delta \to \text{ConditionalExpression}\left[\text{ArcSin}\left[\frac{\Delta\omega}{\lambda}\right] + 2\pi\, C[1],\ C[1] \in \text{Integers}\right]\right\}\right\}$

In[100]:=
$\delta 1 := \text{ArcSin}\left[\frac{\Delta\omega}{\lambda}\right]$

$\delta 2 := \pi - \text{ArcSin}\left[\frac{\Delta\omega}{\lambda}\right]$

In[133]:=
```
vals := {λ → 1, Δω → 0.5}
Plot[U[δ] /. vals, {δ, 0, 8 π},
 Epilog → {
   Red, Table[Disk[{δ1 + 2 π n, U[δ1 + 2 π n]}, .25] /. vals, {n, 0, 0}],
   Green, Table[Disk[{δ2 + 2 π n, U[δ2 + 2 π n]}, .25] /. vals, {n, 0, 0}]
  }]
```

Out[134]=


In[138]:=
```
(* Gleft *)
```

In[168]:=
```
ΔE = U[δ1] - U[δ2] // Simplify
```

Out[168]= $-\pi\,\Delta\omega + 2\sqrt{1 - \frac{\Delta\omega^2}{\lambda^2}}\,\lambda + 2\,\Delta\omega\,\text{ArcSin}\left[\frac{\Delta\omega}{\lambda}\right]$

1/

In[169]:=
```
τ1 = -ddU /. {δ → δ1}
```

Out[169]= $\sqrt{1 - \frac{\Delta\omega^2}{\lambda^2}}\,\lambda$

In[181]:= $\frac{\text{rleft}}{\text{rright}}$ // FullSimplify

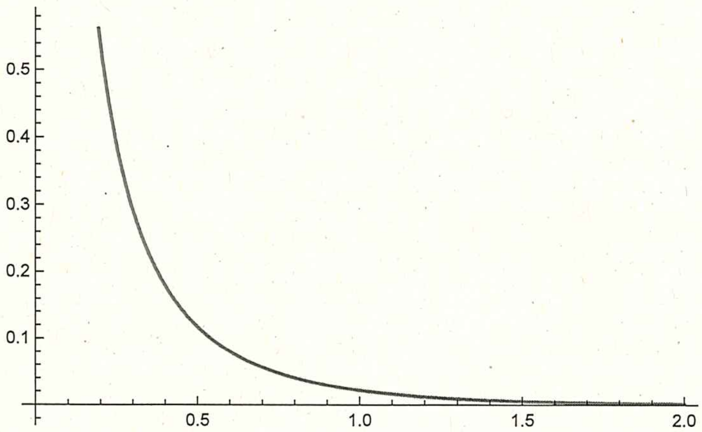
Out[181]= $e^{\frac{2\pi\Delta\omega}{D}}$

In[182]:= (* symmetric case: no frequency mismatch *)
rleft /. {Δω → 0} // PowerExpand

Out[182]= $\frac{e^{-\frac{2\lambda}{D}}}{2\pi\lambda}$

In[185]:= Plot[rleft /. {Δω → 0, D → 1} // PowerExpand, {λ, 0, 2}]

Out[185]=



In[194]:= (* giant diffusion *)
rleftEps = rleft /. {Δω → λ (1 - ϵ)} // Simplify

Out[194]= $\frac{e^{-\frac{2\lambda\left(\sqrt{-(-2+\epsilon)\,\epsilon}+(-1+\epsilon)\,\mathrm{ArcCos}[1-\epsilon]\right)}{D}}}{2\pi\sqrt{-(-2+\epsilon)\,\epsilon\,\lambda^2}}$

In[198]:= Series[rleftEps, {ϵ, 0, 1}] // Normal // PowerExpand

Out[198]= $-\frac{2\epsilon}{3 D\pi}+\frac{1}{2\sqrt{2}\,\pi\sqrt{\epsilon}\,\lambda}+\frac{\sqrt{\epsilon}}{8\sqrt{2}\,\pi\lambda}$

In[199]:= (* → not meaningful [Kramers escape rate theory not applicable] *)

The Kuramoto model of N coupled oscillators.

$$\dot{\varphi}_i = \omega_i - \frac{\lambda}{N}\sum_{k=1}^{N} \sin(\varphi_i - \varphi_j), \quad \lambda > 0.$$

($\Rightarrow$ all-to-all coupling).

$Z_k = \exp i\varphi_k$ $\equiv$ complex oscillator variable.

$$\overline{Z} = \frac{1}{N}\sum_{k=1}^{N} Z_k = r \exp i\psi.$$

$r = |\overline{Z}|$ $\equiv$ order parameter

$\psi = \arg \overline{Z}$ $\equiv$ global phase.

$\Rightarrow$ $\dot{\varphi}_i = \omega_i - \lambda r \sin(\psi - \varphi_i)$.

$\equiv$ single oscillator coupled to mean-field

Let's consider thermodynamic limit $N \to \infty$: with some $p(\omega)$.

- Suppose $r > 0$, then ψ well-def'd.

$\Rightarrow$ $\dot{\psi} = \omega_0$

$\Rightarrow$ $\Delta\omega = \omega - \omega_0$ $\Rightarrow$ $\dot{\varphi} = \Delta\omega + \omega_0 - \lambda r \sin(\varphi - \psi)$.

- Two cases:

(i) $|\Delta\omega| < \lambda r$:

φ phase-locks to $|\bar{z}|$
with phase-lag

$\Delta\varphi = \sin^{-1}(\Delta\omega/(\lambda r))$ $\Rightarrow$

$p_s^*(\varphi|\psi) = \delta(\varphi-\psi-\delta\varphi)$

(ii) $|\Delta\omega| > \lambda r$:

φ displays phase-drift wrt ψ $\Rightarrow$

$p_u^*(\varphi|\psi) \sim \dot{\varphi}-\omega_0 = \Delta\omega + \lambda r \sin(\Delta\varphi)$.

$\Rightarrow$ We obtain self-consistency equation for r:

$$r = |\bar{z}| = \left|\int d\omega \; p(\omega) \cdot p(\varphi|\psi) \exp i\varphi\right|$$

$$= \Big| \int_{|\Delta\omega|<\lambda r} d\omega \; p(\omega) \cdot p_s^*(\varphi|\psi) \exp i\varphi$$

$$+ \int_{|\Delta\omega|>\lambda r} d\omega \; p(\omega) \, p_u^*(\varphi|\psi) \exp i\varphi \Big|.$$

$\Rightarrow$ Solve implicit eqn. for r.

Special Case:

$$p(\omega) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (\omega-\omega_0)^2} \equiv \text{Lorentzian}$$

- location ω_0
- width at half max. 2γ.