Itô vs. Stratonovich

- Q: Given an ODE, e.g.

$$\dot{x} = f(x):$$

What does it mean?

A: Constructive approach: estimate ~~get~~ $x_i = x(i\,dt)$, numerically, then let $dt \to 0$.

- Possibility no 1: explicite Euler scheme

$$x_i = x_{i-1} + f(x_{i-1}) \cdot dt.$$

- Possibility no 2: implicite

$$x_i = x_{i-1} + f(x_i)\,dt.$$

- Possibility no 3: mixed

$$x_i = x_{i-1} + \tfrac{1}{2}[f(x_{i-1}) + f(x_i)]\,dt.$$

$\Rightarrow$ For deterministic ODEs different schemes differ only in terms of numerical performance, stability, etc. Not in the limit result

- What about SDEs ?

$$\dot{x} = f(x) + \sqrt{2D(x)}\,\xi$$

$$\langle \xi(t)\,\xi(t') \rangle = \delta(t-t').$$

- Explicite Euler scheme:

$$x_i = x_{i-1} + f(x_{i-1})\,dt + \sqrt{2D(x_{i-1})}\,\mathcal{N}\cdot\sqrt{dt}.$$

$\equiv$ normally distributed normal variable mean 0, variance 1

$\equiv$ Wiener increment.

$\longrightarrow$ Ito

- Mixed scheme:

$$x_i = x_{i-1} + \tfrac{1}{2}\left[f(x_{i-1}) + f(x_i)\right]dt + \tfrac{1}{2}\left[\sqrt{2D(x_{i-1})} + \sqrt{2D(x_i)}\right]\mathcal{N}\cdot\sqrt{dt}.$$

$\longrightarrow$ Stratonovich

$$x_i = x_{i-1} + f(x_{i-1})\cdot dt + O(dt^{3/2})$$
$$+\, g(x_{i-1})\cdot\mathcal{N}\cdot\sqrt{dt}$$
$$+\, g'(x_{i-1})\cdot g(x_{i-1})\cdot\mathcal{N}^2\cdot dt + O(dt^{3/2})$$

$\Rightarrow \langle \mathcal{N}^2 \rangle = 1 \neq 0$

$\Rightarrow$ Itō vs. Stratonovich interpretation give different drift terms.

Wrap-up

- Stating a nonlinear SDE without specifying its interpretation does not make sense (it's like writing a text and not specifying in which language it was written)

  $\longrightarrow$ Many more interpretations:
  - Isothermal
  - alpha-calculus by Lau & Lubensky PRE

- A Stratonovich SDE $\dot{x} = f(x) + g(x)\xi$ can be rewritten as Itō SDE $\dot{x} = f(x) + g(x)\xi + \frac{1}{2} g' g$ and vice versa. Use interpretation most suitable for a given task.

Itô chain rule

(I) $\dot{x}_k = f_k + g_{kl}\,\xi_l$.

$$\langle \xi_k(t_1)\,\xi_l(t_2)\rangle = \delta_{kl}\,\delta(t_1-t_2)$$

$y = y(\underline{x})$.

$$\dot{y} = \frac{\partial y}{\partial x_j}\dot{x}_j + \frac{1}{2}\frac{\partial^2 y}{\partial x_k \partial x_l} g_{km} g_{lm}.$$

Switching between Itô & Stratonovich

(S) $\dot{x}_k = f_k + g_{kl}\,\xi_l$.

(I) $\dot{x}_k = f_k + g_{kl}\,\xi_l + \frac{1}{2}\frac{\partial g_{kl}}{\partial x_m} g_{ml}$.

Fokker-Planck equation

$$\dot{P} = \frac{\partial}{\partial x_k}\Big[-\big(\underbrace{f_k}_{\text{det. drift}} + \underbrace{\alpha \frac{\partial g_{kl}}{\partial x_m} g_{ml}}_{\text{noise induced drift}}\big)P + \underbrace{\frac{1}{2}\frac{\partial}{\partial x_m}(g_{kl}\,g_{ml})P}_{\text{diffusion}}\Big]$$

$\alpha = 0$: Itô

$\alpha = 1/2$: Stratonovich

$\alpha = 1$: Iso-thermal

Lau-Lubensky PRE

- finance loves Ito: not looking back into the past.

- Physics: often Stratonovich used, often <u>Wong-Zakai-theorem</u> applies:

<u>Theorem</u> (Wong-Zakai): If $\dot{x} = f(x) + g(x)\,\xi$ is an SDE with <u>colored</u> noise of <u>finite</u> correlation time T, then taking the limit T will yield a Stratonovich SDE with Gaussian white noise ξ.

Example: rotational diffusion in 2D.

$$\dot{\varphi} = \xi .$$

$$e_1 = \begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix}, \quad e_2 = \begin{pmatrix}-\sin\varphi \\ \cos\varphi\end{pmatrix}.$$

SO(2): $\frac{d}{dt}[e_1, e_2] = [e_1, e_2]\cdot \xi \cdot \begin{pmatrix}0 & -1 \\ +1 & 0\end{pmatrix}$

$$(S) \quad \begin{cases} \dot{e}_1 = \xi\, e_2 \\ \dot{e}_2 = -\xi\, e_1 \end{cases}$$

$$\Downarrow$$

$$(I) \quad \begin{cases} \dot{e}_1 = \xi\, e_2 + \underbrace{\tfrac{1}{2}\langle \xi \frac{\partial e_2}{\partial e_1}\cdot \dot{e}_1 \rangle}_{0} + \underbrace{\tfrac{1}{2}\langle \xi \frac{\partial e_2}{\partial e_2}\cdot \dot{e}_2 \rangle}_{\frac{1}{2}(2D_{rot})\cdot(-e_1)} \\ \quad = \xi\, e_2 + \cancel{\xi} - D_{rot}\, e_1 \\ \dot{e}_2 = -\xi\, e_1 - D_{rot}\, e_2 . \end{cases}$$

Exercise:

Run simulations for

(I) $\dot{e}_1 = \xi e_2 - D_{rot}\, e_1$
$\dot{e}_2 = -\xi e_1 - D_{rot}\, e_2$

and

(I) $\dot{e}_1 = \xi e_2$
$\dot{e}_2 = -\xi e_1 .$

Extended example:

Persistent random walk (2D).

$$\underline{\dot{r}} = v_0 \underline{e}_1$$

$$\underline{e}_1 = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \quad \dot{\varphi} = \xi, \quad \langle \xi(t_1)\xi(t_2)\rangle = 2D_{rot}\,\delta(t_1-t_2)$$

Proposition

$$C(t) = \langle \underline{e}_1(0)\cdot\underline{e}_1(t)\rangle = \exp(-D_{rot}\,t)$$

$$t_p = \frac{1}{D_{rot}} \equiv \text{persistence time}$$

$$l_p = v_0 t_p \equiv \text{persistence length}$$

Proof:

$$\frac{d}{dt}C(t) = \langle \underline{e}_1(0)\,\dot{\underline{e}}_1(t)\rangle$$

$$= \langle [\underline{e}_1(0)\cdot\underline{e}_1(t)][\underline{e}_1(t)\cdot\dot{\underline{e}}_1(t)]\rangle$$
$$+ \langle [\underline{e}_1(0)\cdot\underline{e}_2(t)][\underline{e}_2(t)\cdot\dot{\underline{e}}_1(t)]\rangle.$$

Ito calculus: factors independent

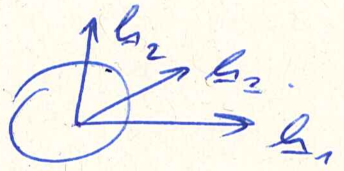
$$= \langle \underline{e}_1(0)\cdot\underline{e}_1(t)\rangle\cdot\langle\underline{e}_1(t)\cdot\dot{\underline{e}}_1(t)\rangle.$$
$$+ \langle \underline{e}_1(0)\cdot\underline{e}_2(t)\rangle\langle\underline{e}_2(t)\cdot\dot{\underline{e}}_1(t)\rangle$$

$$= C(t)\cdot(-D_{rot}) + \sqrt{1-C^2}\cdot\underbrace{\langle\xi\rangle}_{=0}.$$

$$= -D_{rot}\,C(t).$$

q.e.d.

Rotational diffusion in 3D



$$D_{rot} = \frac{k_B T}{8\pi \eta r^3}.$$

spherical coordinates

$$\underline{h}_3 = (\cos\psi, \sin\psi\cos\theta, \sin\psi\sin\theta).$$

$$\underline{h}_1 = (\cos\varphi, \sin\varphi, 0)$$

$$\underline{h}_2 = \underline{h}_3 \times \underline{h}_1.$$

Frenet-Serret equations

$$(S)\begin{cases} \dot{\underline{h}}_3 = \xi_2 \underline{h}_1 - \xi_1 \underline{h}_2 \\ \dot{\underline{h}}_1 = -\xi_2 \underline{h}_3 + \xi_3 \underline{h}_2 \\ \dot{\underline{h}}_2 = -\xi_1 \underline{h}_3 - \xi_3 \underline{h}_1. \end{cases}$$

$\Rightarrow$

(S) $$\dot{\psi} = +\sin\varphi\,\xi_1 + \cos\varphi\,\xi_2.$$

$\Rightarrow$ (I)

$$\dot{\psi} = \underbrace{\sin\varphi\,\xi_1 + \cos\varphi\,\xi_2}_{\text{equivalent to } \xi(t)} + D_{rot}\cot\psi.$$

equivalent to $\xi(t)$ gaussian white noise with $\langle \xi(t_1)\xi(t_2)\rangle = 2D_{rot}\,\delta(t_1 - t_2)$

$$\boxed{\dot{\psi} = D_{rot}\cot\psi + \xi}$$

Steady state distribution $P(\varphi)$ must be isotropic.

- $P(\underline{e}_3 \in dA) = \frac{dA}{4\pi}$, area element dA.

- spherical cap. $A = 2\pi \cdot h$
  $dA = 2\pi \cdot dh$.
  $h = 1 - \cos\varphi$.
  $dh = \sin\varphi \, d\varphi$.

$P(h) = \frac{dh}{2}$
$\Rightarrow$
$P(\varphi) = \frac{\sin\varphi}{2} d\varphi$.

- Alternatively:
  recover potential
  $U(\varphi) = -D_{rot} \ln \sin\varphi$
  $\Rightarrow$
  $D_{rot} \cot\varphi = -\frac{\partial U}{\partial \varphi}$.
  $\Rightarrow$
  $P(\varphi) \sim \exp + \frac{D_{rot} \ln \sin\varphi}{D_{rot}}$
  $\sim \sin\varphi$.