Synchronization

Active oscillators:

- Van-der-Pol:

$$m\ddot{x} + \gamma(2A_0^2 - x^2)\dot{x} + kx = 0.$$

- Hopf

$$\dot{z} = i\omega_0 z + \mu(A_0^2 - |z|^2)z.$$

- Phase oscillator

$$\dot{\varphi} = \omega_0.$$

Two coupled oscillators.

$$\dot{\varphi}_L = \omega_L + C(\varphi_L - \varphi_R)$$
$$\dot{\varphi}_R = \omega_R + C(\varphi_R - \varphi_L).$$

$$\delta = \varphi_L - \varphi_R.$$

$\Rightarrow$ $C(\delta) = C(\delta + 2\pi) \equiv$ coupling function

$$C(\delta) = \sum_n C_n' \cos(n\delta) + C_n'' \sin(n\delta).$$

$\Rightarrow$

$$\dot{\delta} = \Delta\omega + 2\sum_n C_n'' \sin n\delta$$

$\Rightarrow$ only odd coupling terms contribute to synchronization.

Often C dominated by first Fourier mode.

$$\boxed{\dot{\delta} = \Delta\omega - \lambda \sin\delta}$$

$\equiv$ Adler equation ($\lambda = -2C_1''$).

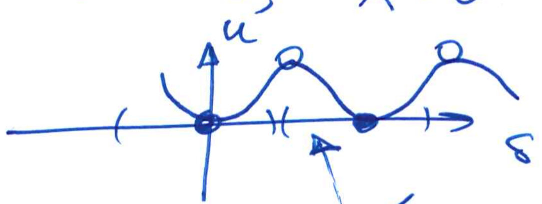
- fixed points:

  If $\Delta\omega = 0$: $\delta^* = 0$, $\delta^* = \pi$.

  generally, $\delta^* = \sin^{-1}(\Delta\omega/\lambda)$ if $|\Delta\omega| < |\lambda|$

- Stability? $\underbrace{\gamma\dot{\delta} = -\dfrac{\partial u}{\partial \delta}}_{\text{over-damped motion.}}$, $u = \underbrace{-\gamma\Delta\omega\,\delta - \gamma\lambda\cos\delta}_{\text{eff. potential.}}$

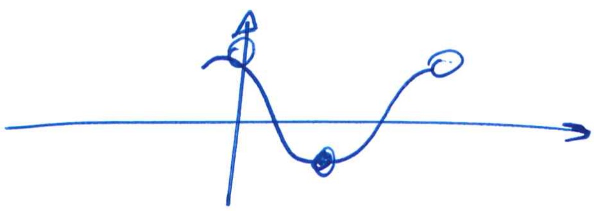
$\Delta\omega = 0$, $\lambda > 0$



basin of attraction.

in-phase synch.

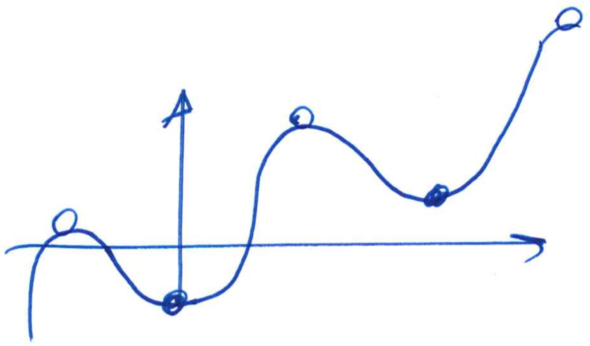
$\Delta\omega = 0$, $\lambda < 0$



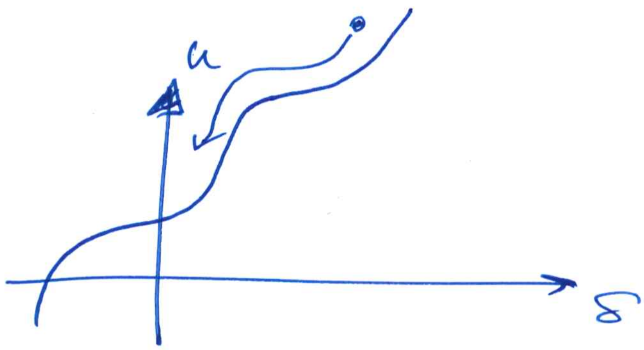
anti-phase synch.

$0 < \Delta\omega < \lambda$

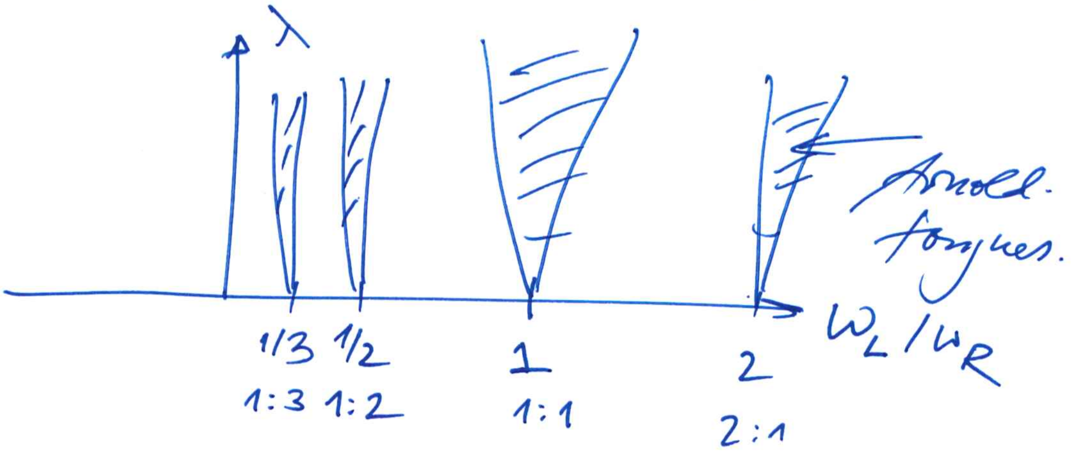
phase-lag between both oscillators.



$|\Delta\omega| > \lambda$: phase-drift (no synchr.).



- If $\omega_L : \omega_R \approx n:m$, $n, m \in \mathbb{N}$. There can be n:m-synchronisation



Synchronization in the presencence of noise

$$\dot{\delta} = \Delta\omega - \lambda \sin \delta + \xi.$$

$\xi(t) \equiv$ Gaussian white noise.

$\langle \xi(t) \rangle = 0.$

$\langle \xi(t)\xi(t') \rangle = 2D\,\delta(t-t').$

$[D] = [\frac{1}{s}].$

$$\gamma \dot{\delta} = -\frac{\partial u}{\partial \delta} + \xi.$$

$\xi(t) \equiv$ Gaussian white noise

$\langle \xi(t) \rangle = 0.$

$\langle \xi(t)\xi(t') \rangle = 2D'\,\delta(t-t').$

$$D' = \frac{kT_{eff}}{\gamma}\gamma^2.$$

formally similar to temperature, yet can be due to active processes.

What is the effect of noise?

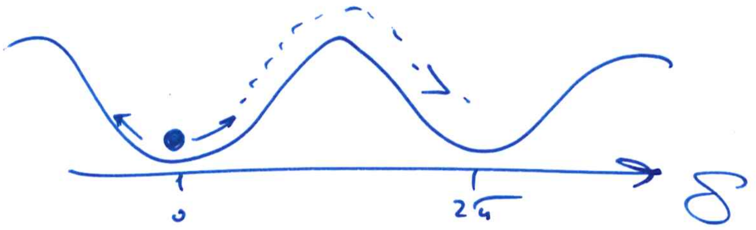
$\Rightarrow$ Synchronization perturbed

- δ fluctuates around δ^*
- occasionally phase slips: $\delta \to \delta \pm 2\pi$.

$\Rightarrow$ Steady-state probability distribution formally equiv. to Boltzmann distrib.

$$p(\delta) \sim \exp\left(-\frac{U(\delta)}{kT_{eff.}}\right) \quad \text{for } \Delta\omega = 0.$$

$$= \frac{1}{2\pi I_0(\lambda/D)} \exp\left(\frac{\lambda}{D}\cos\delta\right).$$



phase-slips:

$\delta \longrightarrow \delta + 2\pi$: with rate Γ_+

$\delta \longrightarrow \delta - 2\pi$: — " — Γ_-

$$\Gamma_\pm = \frac{D}{4\pi^2} \left| I_{i\Delta\omega/D}(\lambda/D) \right|^{-2} \exp\left(\pm\frac{\Delta\omega}{D}\right)$$

For $\Delta\omega = 0$:

$$\Gamma_+ = \Gamma_- = \begin{cases} \exp(-2\lambda/D)/(2\pi) & D \ll |\lambda| \\ D/(2\pi)^2 & D \gg |\lambda|. \end{cases}$$

$\longrightarrow$ Stratonovich

The Kuramoto model of N coupled oscillators.

$$\dot{\varphi}_i = \omega_i - \frac{\lambda}{N}\sum_{k=1}^{N} \sin(\varphi_i - \varphi_j)., \quad \lambda > 0.$$

($\Rightarrow$ all-to-all coupling).

$z_k = \exp i\varphi_k. \equiv$ complex oscillator variable.

$$\bar{z} = \frac{1}{N}\sum_{k=1}^{N} z_k = r \exp i\psi.$$

$r = |\bar{z}| \equiv$ order parameter

$\psi = \arg \bar{z} \equiv$ global phase.

$\Rightarrow$ $\dot{\varphi}_i = \omega_i - \lambda r \sin(\psi - \varphi_i).$

$\equiv$ single oscillator coupled to mean-field.

Let's consider thermodynamic limit $N \to \infty$: with some $p(\omega)$.

- Suppose $r > 0$, then ψ well-def'd.

$\Rightarrow$

$\Rightarrow$ $\dot{\psi} = \omega_0$

$\Delta\omega = \omega - \omega_0 \Rightarrow \dot{\varphi} = \Delta\omega + \omega_0 - \lambda r \sin(\varphi - \psi).$

- Two cases:

(i) $|\Delta\omega| < \lambda r$:

φ phase-locks to $|\bar{z}|$

with phase-lag

$\Delta\varphi = \sin^{-1}(\Delta\omega/(\lambda r))$. $\Rightarrow$

$p_s^*(\varphi|\psi) = \delta(\varphi-\psi-\Delta\varphi)$

(ii) $|\Delta\omega| > \lambda r$:

φ displays phase-drift wrt ψ $\Rightarrow$

$p_u^*(\varphi|\psi) \sim \dot{\varphi} - \omega_0 = \Delta\omega + \lambda r \sin(\Delta\varphi)$.

$\Rightarrow$ We obtain self-consistency equation for r:

$$r = |\bar{z}| = \left| \int d\omega \; p(\omega) \cdot p(\varphi|\psi) \exp i\varphi \right|$$

$$= \Big| \int_{|\Delta\omega| < \lambda r} d\omega \; p(\omega) \cdot p_s^*(\varphi|\psi) \exp i\varphi$$

$$+ \int_{|\Delta\omega| > \lambda r} d\omega \; p(\omega) \, p_u^*(\varphi|\psi) \exp i\varphi \Big|.$$

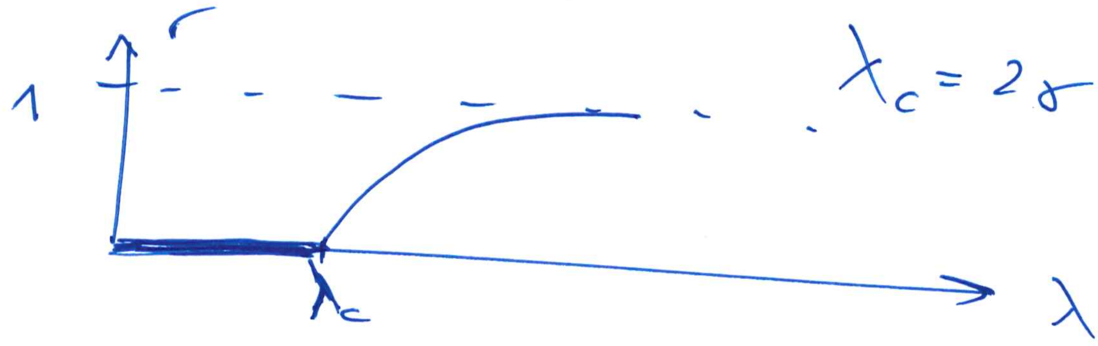
$\Rightarrow$ Solve implicit eqn. for r.

Special case:

$$p(\omega) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (\omega-\omega_0)^2}$$ $\equiv$ Lorentzian

- location ω_0
- width at half max. 2γ.

$$\Rightarrow r = \begin{cases} 0 & \lambda < \lambda_c \\ \sqrt{1 - \dfrac{\lambda_c}{\lambda}} & \lambda > \lambda_c \, ; \end{cases}$$



$\lambda_c = 2\gamma .$