§10

• TODO: start with crayfish

Stochastic resonance

Motivational example:
Ice ages

- $x(t) \equiv$ global ice volume
  $|\tilde{x}(\omega)|^2$: peak at $\Omega = \frac{2\pi}{T}$, $T \approx 10^5 y$.
- external driving?
  eccentricity of earth orbit oscillates with $T = 0.96 \cdot 10^5 y$,
  induces $O(0.3K)$ temperature variation by change in irradiation.
  $\longrightarrow$ too weak to account for $\sim 10K$ temperature change during ice age.
- add noise!

- Beneficial effect of noise?
$\Rightarrow$ detect weak oscillatory stimuli.

two-state (telegraph) process.

$x = -1 \longleftrightarrow x = +1$

$r_- : \quad n_- \qquad \longrightarrow \qquad n_+$

$\qquad p_- \qquad\qquad\qquad p_+$

$r_+ : \quad n_- \qquad \longleftarrow \qquad n_+$

$\qquad p_- \qquad\qquad\qquad p_+$

$$\dot n_+ = -\dot n_- = r_- n_- - r_+ n_+ .$$

$$= r_- - (\underbrace{r_+ + r_-}_{r}) n_+ .$$

formal solution

$$n_+(t) = \left(\exp - \int_{t_0}^{t} dt' \, r\right) n_+(t_0)$$

$$+ \exp\left(-\int_{t_0}^{t} dt' r\right) . \int_{t_0}^{t} dt' \, r_-(t') . \exp \int_{t_0}^{t'} r \, dt''$$

$\longrightarrow$ skip? next time

Note

$$\frac{d}{dt}\left[\left(\exp \int^t r\right) n_+\right] = r_- \exp \int^t dt' \, r .$$

check:

- $n_+(t_0) = 1 \cdot n_+(t_0) + 1 \cdot 0 .$
- $\dot n_+(t_0) = -r \; n_+(t) +$

$$\exp\left(-\int_{t_0}^{t} dt' r\right) . r_-(t) .$$

$$\exp\left(+\int_{t_0}^{t} dt'' r\right) .$$

$$= - r n_+(t) + r_-$$

$p_+ = \frac{n_+}{n}$

Next: autocorrelation function

$\longrightarrow$

power spectrum.

$$\langle x(t) \rangle = x_+ n_+(t) + x_- n_-(t)$$
$$= n_+ - n_- \quad \text{for } x_\pm = \pm 1$$

$$\langle x(t) \cdot x(t+T) \mid x_0, t_0 \rangle =$$

Markov property

$$= \sum_{s,s'} s s' \, P(s, t+T \mid s', t) \, P(s', t \mid x_0, t_0)$$

$$= n_+(t+T \mid +1, t) \cdot n_+(t \mid x_0, t_0)$$
$$- n_+(t+T \mid -1, t) \cdot n_-(t \mid x_0, t_0)$$
$$- n_-(t+T \mid +1, t) \cdot n_+(t \mid x_0, t_0)$$
$$+ n_-(t+T \mid -1, t) \cdot n_-(t \mid x_0, t_0).$$

- use $n_- = 1 - n_+$
- substitute above result
- limit $t_0 = -\infty$

$$\langle x(t) x(t+T) \mid x_0, t_0 \rangle =$$
$$\exp(-2 r_0 T) + 2 r_1 +$$
$$4 r_1^2 [-\exp(-2 r_0 T) \cos^2(\Omega t - \varphi)$$
$$+ \cos(\Omega(t+T) - \varphi) \cdot \cos(\Omega t - \varphi)].$$

Example: spectral power density of Ornstein-Uhlenbeck process

$$\tau \dot{x} = -x + \xi$$

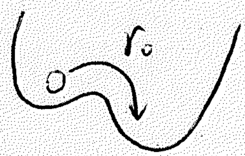
$$(1 + \tau i\omega)\tilde{x}(\omega) = \tilde{\xi}(\omega)$$

$$\begin{aligned} S_x(\omega) &= \langle |\tilde{x}(\omega)|^2 \rangle \\ &= \left| \frac{1}{1+\tau i\omega} \right|^2 \langle |\tilde{\xi}(\omega)|^2 \rangle \\ &= \frac{2D}{(\tau\omega)^2 + 1}. \end{aligned}$$

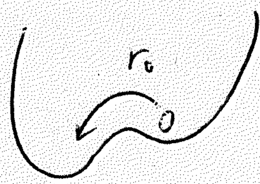
$\equiv$ Lorentzian

$\Rightarrow$ SPA attains maximum
at $\Omega = 2 r_0$

$\Rightarrow$ Intuitive interpretation



$\frac{1}{r_0} \approx \frac{T}{2}$



$\frac{1}{r_0} \approx \frac{T}{2}$.

Biological exple

Mechano-receptors in crayfish

- predator = periodic pressure signal
  turbulence = noise
- sensory detection optimal at physiological noise levels.
- experimental test by playing signal + $\frac{\text{variable}}{\text{noise}}$.
  and recording neuron spiking