Breaking Synchrony by Heterogeneity in Complex Networks

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For networks of pulse-coupled oscillators with complex connectivity, we demonstrate that in the presence of coupling heterogeneity precisely timed periodic firing patterns replace the state of global synchrony that exists in homogenous networks only. With increasing disorder, these firing patterns persist until a critical temporal extent is reached that is of the order of the interaction delay. For stronger disorder the periodic firing patterns cease to exist and only asynchronous, aperiodic states are observed. We derive self-consistency equations to predict the precise temporal structure of a pattern from a network of given connectivity and heterogeneity. Moreover, we show how to design heterogeneous coupling architectures to create an arbitrary prescribed pattern.

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Understanding how the structure of a complex network [1] determines its dynamics is currently in the focus of research in physics, biology, and technology [2]. Pulse-coupled oscillators provide a paradigmatic class of models to describe a variety of networks that occur in nature, such as populations of fireflies, pacemaker cells of the heart, earthquakes, or neural networks [3,4]. Synchronization is one of the most prevalent kinds of collective dynamics in such networks [5]. Recent theoretical studies, which analyze conditions for the existence and stability of synchronous states, have focused on *homogenous* networks with simple topologies, e.g., global couplings [6] or regular lattices [7]. In nature, however, intricately structured and *heterogeneous* interactions are ubiquitous. Previously, aspects of heterogeneity have been studied mostly in globally coupled networks [8–10]. However, for networks with structured connectivity only a few studies exist [11] and it is still an open question how heterogeneity influences the dynamics, in particular synchronization, in such networks.

In this Letter we present an exact analysis of the dynamics of complex networks of pulse-coupled oscillators in the presence of heterogeneity in the coupling strengths. We demonstrate that the synchronous state, that exists in homogenous networks, is replaced by precisely timed periodic firing patterns. The temporal extent of these patterns grows with the degree of disorder. Patterns persist below a critical strength of the disorder beyond which only asynchronous, aperiodic states are observed. We show how this transition is controlled by the interaction delay. Simple criteria for the stability of firing patterns are derived. Furthermore, we present an approach to predict the relative timings of firing events from self-consistency conditions for the phases in a given network. Conversely, a prescribed pattern can be created by designing a heterogeneous coupling architecture in a network of specified connectivity.

Consider a fixed network of N pulse-coupled oscillators that interact via directed connections. A matrix C determines the connectivity of this network, where $C_{ij} = 1$ if a connection from oscillator j to i exists, and $C_{ij} = 0$ otherwise ($C_{ii} \equiv 0$). The number $k_i := \sum_j C_{ij}$ of inputs to every oscillator i is nonzero, $k_i \geq 1$, and no further restriction on the network topology is imposed. In simulations, we illustrate our findings using random graphs in which every directed connection is present with probability p.

The state of an individual oscillator j is represented by a phase variable ϕ_j that increases uniformly in time [3],

$$d\phi_j/dt = 1. \tag{1}$$

Upon crossing the threshold $\phi = 1$ at time t_f an oscillator is instantaneously reset to zero, $\phi_j(t_f^+) = 0$, and a pulse is sent. After a delay time τ, this pulse is received by all oscillators i connected to j (for which $C_{ij} = 1$) and induces an instantaneous phase jump

$$\phi_i((t_f + \tau)^+) = U^{-1}(U(\phi_i(t_f + \tau) + \varepsilon_{ij})). \tag{2}$$

Here, ε_{ij} are the coupling strengths from j to i, which are taken to be either purely inhibitory (all $\varepsilon_{ij} \leq 0$) or purely excitatory (all $\varepsilon_{ij} \geq 0$). The interaction function U is monotonically increasing, $U' > 0$, and represents the subthreshold dynamics of individual oscillators. We consider functions with a curvature of constant sign, i.e., $U''(\phi) > 0$ or $U''(\phi) < 0$ for all ϕ. In simulations, we use $U(\phi) = \ln(1 + (e^b - 1)\phi)/b$, where b parametrizes the curvature of U.

We briefly consider homogenous networks with individual coupling strengths $\varepsilon_{ij} = \hat{\varepsilon}_{ij}$ that are normalized

$$\sum_{j=1}^{N} \hat{\varepsilon}_{ij} = \hat{\varepsilon} \tag{3}$$

such that every oscillator i receives the same total input. Such networks exhibit a synchronous state, defined by

$$\phi_i(t) = \phi(t) \tag{4}$$

for all i. The stability of this state has been determined previously [12] and depends on the interplay between the sign of the coupling (excitatory or inhibitory) and the sign of the curvature of U [13].

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Heterogeneity of the couplings is introduced through

$$\varepsilon_{ij} = \hat{\varepsilon}_{ij} + \Delta\varepsilon J_{ij} C_{ij}/k_i \tag{5}$$

such that in general the total input is different for every oscillator, while the connectivity C_{ij} remains fixed. The matrix J specifies the structure of the heterogeneity. The degree of heterogeneity is quantified by the disorder strength $\Delta\varepsilon$. In numerical simulations, we take the homogenous part of the coupling strength to be $\hat{\varepsilon}_{ij} = \hat{\varepsilon} C_{ij}/k_i$ and independently draw the matrix elements J_{ij} from the uniform distribution on $[-1, 1]$.

The synchronous state (4) of homogenous networks ($\Delta\varepsilon = 0$) is replaced by periodic firing patterns in the presence of heterogeneity ($\Delta\varepsilon > 0$), cf. Fig. 1. After a discontinuous transition at a certain critical disorder $\Delta\varepsilon_c$ these patterns disappear and aperiodic, asynchronous states are observed [14].

The firing patterns emerging for $\Delta\varepsilon < \Delta\varepsilon_c$ are confined to a subinterval of the phase axis. The extent of a pattern

$$\Delta\phi = \max_i \phi_i - \min_i \phi_i \tag{6}$$

(taking into account wraparound effects at the threshold) increases as the disorder strength $\Delta\varepsilon$ is increased, whereas the firing order is determined by the structure J of the heterogeneity. If $\Delta\varepsilon \geq \Delta\varepsilon_c$ the periodic pattern
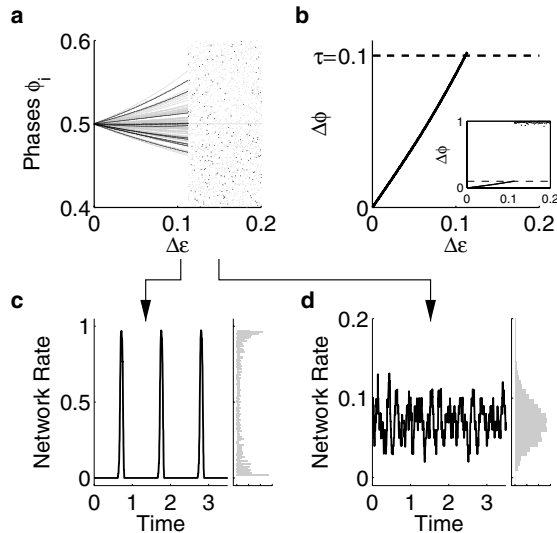
is replaced either directly by an asynchronous state (Fig. 1), or it reaches this state through a sequence of states consisting of several separated clusters of firing patterns (not shown).

To understand the origin of this transition, we note that the critical disorder $\Delta\varepsilon_c$ is directly related to the critical pattern extent $\Delta\phi_c$ before breakdown [cf. Fig. 1(b)]. A periodic pattern is always observed for $\Delta\phi \leq \tau$. For globally connected networks, the transition to an asynchronous state occurs at

$$\Delta\phi_c^{\text{glob}} = \tau \tag{7}$$

(see Fig. 2). If the system is initialized with a pattern of extent $\Delta\phi > \Delta\phi_c^{\text{glob}}$, some oscillators, which we term critical, send pulses that affect oscillators that have not fired within the period considered, causing divergence from the periodic state. In networks that are not globally connected, the firing patterns may persist up to a critical extent $\Delta\phi_c > \Delta\phi_c^{\text{glob}}$ that depends on the specific network structure. For large random networks, we estimate an upper bound $\Delta\phi_c^{\text{max}}$ for $\Delta\phi_c$ by assuming that the firing times (i.e., the phases) within the pattern are uniformly and independently distributed in an interval of length $\Delta\phi$ [15]. We first calculate the probability

$$P' = 1 - \frac{\tau}{\Delta\phi} - \frac{1}{k+1}\left[1 - \left(\frac{\tau}{\Delta\phi}\right)^{k+1}\right] \tag{8}$$

that a given oscillator i is critical, i.e., that of the approximately $k = pN$ oscillators j receiving input from i, at least one satisfies $\phi_i - \phi_j > \tau$. In a second step, we calculate the probability $P = 1 - (1 - P')^N$ that at least one of the N oscillators is critical. Setting $P = 1$ guarantees
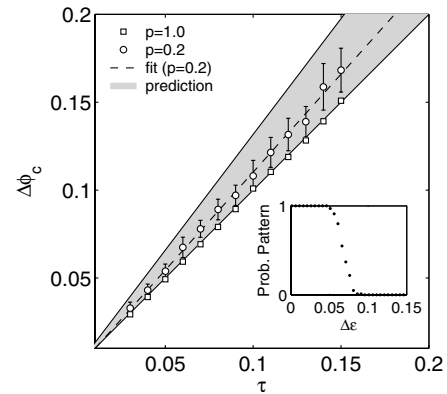

FIG. 1. Heterogeneities in a random network ($N = 100$, $p = 0.4$, $b = 2$, $\hat{\varepsilon} = -0.1$, $\tau = 0.1$) induce desynchronization followed by a transition to an asynchronous, aperiodic state. (a) Relative phases (ten selected phases shown in black) for different disorder strengths $\Delta\varepsilon$ (fixed J). (b) Increase of the extent $\Delta\phi$ of the pattern shown in (a), until the pattern disappears at $\Delta\phi_c \approx \tau$ (dashed line at $\Delta\phi = \tau$). Inset: Onset of asynchronous state ($\Delta\phi \approx 1$) for large disorder. (c),(d) Instantaneous network rate of (c) a periodic ($\Delta\varepsilon = 0.10$) and (d) an aperiodic ($\Delta\varepsilon = 0.15$) state in real time computed using a triangular sliding time window. Gray histograms on the right of (c) and (d) indicate rate distribution [excluding peak at zero network rate in (c)].

FIG. 2. The critical extent of a pattern $\Delta\phi_c$ as a function of the interaction delay τ ($N = 100$, $b = 2$, $\hat{\varepsilon} = -0.1$). Each data point is obtained for a different heterogeneity J in fully connected networks ($\square$) and random networks ($\bigcirc$) of different connectivity C with $p = 0.2$. For the random network, data are averaged over 20 trials (dashed line: fit). Gray area indicates the theoretical second-order prediction of the transition zone between $\Delta\phi_c^{\text{glob}}$ and $\Delta\phi_c^{\text{max}}$. Inset: Probability that a periodic pattern emerges from synchronous initial conditions for a given disorder $\Delta\varepsilon$ (histogram of 100 random heterogeneities and connectivities, $p = 0.2$, $\tau = 0.1$).

the existence of a critical oscillator and results in an implicit formula for $\Delta\phi_c^{\max}$ in terms of N, p and τ. To second order in $\Delta\phi - \tau$ we obtain

$$\Delta\phi_c^{\max} = (1 + \sqrt{2/k})\tau \qquad (9)$$

as an approximate upper bound on the critical pattern extent $\Delta\phi_c$. Numerical data are in good agreement with our theoretical predictions $\Delta\phi_c^{\text{glob}}$ and $\Delta\phi_c^{\max}$ (cf. Fig. 2).

To examine the stability of periodic firing patterns in the presence of heterogeneous couplings ε_{ij}, we consider a pattern

$$\phi_i(zT) = \phi_0 + \Delta\phi_{i,0}, \qquad z \in \mathbb{Z}, \qquad (10)$$

of period T where ϕ_0 represents a common reference phase and $\Delta\phi_{i,0}$ defines the relative phase shift of oscillator i within the pattern. For simplicity, we order the firing events according to $\Delta\phi_{i,1} > \Delta\phi_{i,2} > \cdots > \Delta\phi_{i,k_i}$ where $\Delta\phi_{i,n}$ denotes the phase shift of the oscillator from which oscillator i receives its nth signal within a given period. For example, if j' fires first of all oscillators connected to i, then $\Delta\phi_{i,1} = \Delta\phi_{j',0}$, cf. Eq. (10).

By definition, all phases increase uniformly in time except for two kinds of discrete events: sending and receiving pulses. We follow the dynamics of a periodic state, $\phi_i(0) = \phi_i(T)$ for all i, event by event (cf. [12]) starting with a reference phase ϕ_0 chosen such that all oscillators have fired but not yet received the generated pulses within a given period. The collective period of the unperturbed firing pattern is given by

$$T = \tau + 1 + \Delta\phi_{i,0} - \Delta\phi_{i,k_i} - \sigma_{i,k_i}^{\Delta\phi}, \qquad (11)$$

where $\sigma_{i,0}^{\chi} := \tau$ and

$$\sigma_{i,n}^{\chi} := U^{-1}(U(\sigma_{i,n-1}^{\chi} + \chi_{i,n-1} - \chi_{i,n}) + \varepsilon_{i,n}) \qquad (12)$$

(here with $\chi = \Delta\phi$) are the recursively defined phases of oscillator i right after having received the first n of its k_i inputs ($\varepsilon_{i,1}, \ldots, \varepsilon_{i,k_i}$, ordered accordingly) in a given period.

Adding small perturbations $\delta_{i,0}(0) =: \delta_{i,0}$ satisfying

$$\max_i |\delta_{i,0}| < \min_{i \neq j} |\Delta\phi_{i,0} - \Delta\phi_{j,0}|/2 \qquad (13)$$

to the phases of the oscillators i before firing ensures preservation of the ordering of the firing events: Thus, denoting initial phases as $\phi_i(0) = \phi_0 + \Delta\varphi_{i,0}$, where $\Delta\varphi_{i,0} := \Delta\phi_{i,0} + \delta_{i,0}$, and sorting these relative phase shifts, $\Delta\varphi_{i,1} > \Delta\varphi_{i,2} > \cdots > \Delta\varphi_{i,k_i}$, it follows that $\Delta\varphi_{i,n} = \Delta\phi_{i,n} + \delta_{i,n}$ for all i and for all n. Following the evolution of the perturbed state event by event we obtain an expression for the time $T_i = \frac{\tau}{2} + 1 - \Delta\varphi_{i,k_i} - \sigma_{i,k_i}^{\Delta\varphi}$ to the next firing of i. This leads to a nonlinear period-T map

$$\delta_{i,0}(T) = T - \left(T_i + \frac{\tau}{2} + \Delta\phi_{i,0}\right) = \delta_{i,k_i} - \sigma_{i,k_i}^{\Delta\phi} + \sigma_{i,k_i}^{\Delta\varphi} \qquad (14)$$

of the perturbations, where we used definition (12) with $\chi = \Delta\phi$ and $\chi = \Delta\varphi$. Approximating

$$\sigma_{i,k_i}^{\Delta\varphi} \doteq \sigma_{i,k_i}^{\Delta\phi} + \sum_{n=1}^{k_i} (\delta_{i,n-1} - \delta_{i,n})p_{i,n-1} \qquad (15)$$

to first order in $\delta_{i,0}$, where $p_{i,n} := \prod_{l=n}^{k_i-1} (U'(\sigma_{i,l}^{\Delta\phi} + \Delta\phi_{i,l} - \Delta\phi_{i,l+1})/U'(\sigma_{i,l+1}^{\Delta\phi}))$ for $n < k_i$ and $p_{i,k_i} = 1$, yields

$$\delta_{i,0}(T) \doteq \sum_{j=1}^{N} M_{ij}\delta_{j,0}. \qquad (16)$$

Here, the matrix M_{ij} has diagonal elements $M_{ii} = p_{i,0}$, and the nonzero, off-diagonal elements are $M_{ij} = p_{i,n} - p_{i,n-1}$, where j is the index of the oscillator sending the nth signal to i (i.e., $\Delta\phi_{j,0} = \Delta\phi_{i,n}$). Time translation invariance implies $\sum_j M_{ij} = 1$ for all i. Furthermore, if either $U'' > 0$ and $\hat{\varepsilon} > 0$ or $U'' < 0$ and $\hat{\varepsilon} < 0$, then $p_{i,l} < p_{i,m} < 1$ for $l < m$ which guarantees that all $M_{ij} \geq 0$. Under these conditions, it is straightforward to show that $\max_i |\delta_{i,0}(T)| \leq \max_i |\delta_{i,0}(0)|$, thus implying stability of the firing pattern.

Numerical simulations indicate that firing patterns are determined by the network topology and the structure of the heterogeneity. How does heterogeneity in a complex network control the precise timing of pulses that constitute a pattern? In the following we analytically predict the pattern resulting from the coupling architecture in a network of known connectivity. Assume that the given couplings ε_{ij} induce a pattern (10) of period T. A change in the coupling matrix $\varepsilon'_{ij} = \varepsilon_{ij} + \delta\varepsilon_{ij}$ below the critical disorder leads to a new periodic state $\phi'_i(zT') = \phi_i(0) + \delta\phi_{i,0}$ of similar period T' that is characterized by the phase shift $\boldsymbol{\delta\phi} = (\delta\phi_{1,0}, \ldots, \delta\phi_{N,0})$. Defining $\Delta\Phi_{i,n} := \Delta\phi_{i,n} + \delta\phi_{i,n}$, we pick an arbitrary reference oscillator l and adjust time such that its reference phase is $\Delta\Phi_{l,0} = 0$. The period of oscillator l in the network defined by the couplings ε'_{ij} is then given by $T_l = 1 + \tau - \Delta\phi_{l,k_l} - \delta\phi_{l,k_l} - \sigma_{l,k_l}^{\Delta\Phi}$ (where $\delta\phi_{i,j}$ is an ordering of $\delta\phi_{i,0}$ in analogy to the ordering $\Delta\phi_{i,j}$ of $\Delta\phi_{i,0}$). After the time T_l the phase of an oscillator $i \neq l$ is given by

$$\phi'_i(T_l) = \sigma_{i,k_i}^{\Delta\Phi} - \sigma_{l,k_l}^{\Delta\Phi} + \Delta\phi_{i,k_i} - \Delta\phi_{l,k_l} + \delta\phi_{i,k_i} - \delta\phi_{l,k_l}. \qquad (17)$$

Periodicity of this state requires

$$R_i(\boldsymbol{\delta\phi}, \delta\varepsilon) := \phi'_i(T_l) - \phi'_i(0) \overset{!}{=} 0 \qquad (18)$$

for all i, which is an exact, implicit system of algebraic equations for $\boldsymbol{\delta\phi}$. The collective period T' of the pattern is then given by T_l evaluated at the actual phase shifts $\boldsymbol{\delta\phi}$. Expanding R_i and T_l to first order in $\delta\varepsilon_{ij}$ and $\delta\phi_{i,0}$ yields

$$\sum_j \frac{\partial R_i}{\partial \delta\phi_{j,0}}\bigg|_{(0,0)} \delta\phi_{j,0} = -\sum_j \frac{\partial R_i}{\partial \delta\varepsilon_{ij}}\bigg|_{(0,0)} \delta\varepsilon_{ij} \qquad (19)$$

because $R_i(0,0) = \phi_i(T) - \phi_i(0) = 0$ by definition. Thus $\boldsymbol{\delta\phi}$ can be approximated to first order by solving a system of N linear equations. This first-order prediction yields a good approximation of the actual firing pattern for small disorder $\Delta\varepsilon$ [see, e.g., Figs. 3(a) and 3(b)] and reasonably
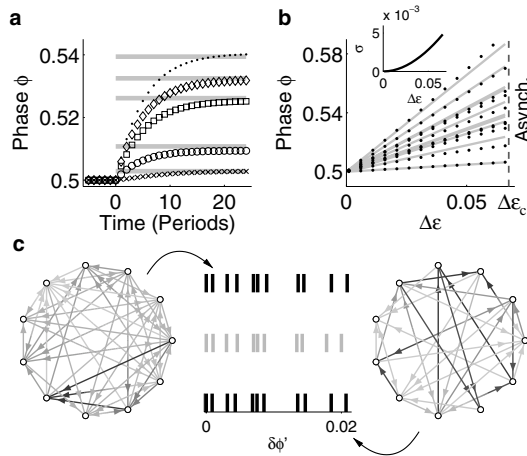
FIG. 3. Predicting firing patterns and designing coupling architectures to create a prescribed pattern ($b = 2$, $\hat{\varepsilon} = -0.1$, $\tau = 0.1$, predictions to first order). (a),(b) Comparison of prediction (gray lines) and simulation (markers) of selected phases of a pattern ($N = 100$, $p = 0.4$). (a) Phases shown period by period for $\Delta\varepsilon = 0.25$. Heterogeneity is instantaneously introduced at time $t = 0$. The attractor is reached asymptotically. (b) Pattern for increasing disorder $\Delta\varepsilon$ (J fixed). Inset: standard deviation of prediction from simulation. (c) Coupling strengths (indicated by shading) in two different heterogeneous networks ($N = 11$, left: $p = 0.6$, right: $p = 0.3$) designed to create a prescribed firing pattern (middle row, gray). Patterns in black are obtained by simulating these designed networks (initialized to the synchronous state).

describes the relative firing times up to $\Delta\varepsilon = \Delta\varepsilon_c$ [cf. Fig. 3(b)].

The above method may be reversed to design heterogeneous coupling architectures in order to create a prescribed pattern defined by $\delta\phi$. Here, we fix the period T' via the reference oscillator l, and solve each of the plane equations i derived by linearizing (18) with $T' = T_l$ for a suitable set of $\delta\varepsilon_{ij}$. Examples of two networks of different connectivity designed to exhibit the same prescribed pattern are shown in Fig. 3(c).

In summary, we have demonstrated that for small coupling disorder the synchronous state is replaced by precisely timed periodic firing patterns. We have shown how to predict the precise timing of pulses from a given coupling architecture, and reversely how to design networks in order to create prescribed patterns. There is a critical disorder strength, which we relate to the interaction delay τ, beyond which only asynchronous, aperiodic states are observed. Similar behavior is expected for other sources of parameter heterogeneity that likewise induce a distribution of effective frequencies in the individual oscillators, such as distributions of delays or interaction functions $U(\phi)$. The continuous transition found for small disorder is very different from that found previously for threshold-induced synchronization where oscillators are split into one synchronous and one asynchronous subpopulation due to heterogeneity [8]. It has been shown recently that periodic firing patterns can be obtained and

even learned in networks with global inhibition and no interaction delays as a perturbation of the asynchronous state [16]. In contradistinction, the patterns discussed in this Letter are not related to (but coexist with) asynchronous states, and emerge from the synchronous state due to heterogeneity. The simplicity of the original synchronous state aided us in clarifying how heterogeneity of networks with a complicated structure controls their precise dynamics. These results are a first step towards understanding and designing the dynamics of real world networks that exhibit complex structure and heterogeneity.

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[14] Numerical simulations start with phases initialized in the synchronous state (4) of the homogenous network. The disorder $\Delta\varepsilon$ is increased (at fixed J) from $\Delta\varepsilon = 0$ in increments of 10^{-3}. For each value of $\Delta\varepsilon$, the phases are evolved for 300 firings of a reference oscillator, plotted, and used as initial conditions for the simulation at the next higher value of $\Delta\varepsilon$.

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